THE PREP

Saint Joseph's Preparatory School

Calculus

Algebra/Pre-Calc Review Packet

IMPORTANT NOTICE:

This packet contains a review of material that you are expected to have mastered. It is strongly recommended that you read this packet in its entirety and **complete all practice problems.** These problems will be considered your first homework assignment of the first quarter. They should be completed in a **MARBLE NOTEBOOK** that you will use throughout the year for your homework assignments**.** We will review this packet to start the school year.

A TEST ON THIS MATERIAL WILL BE GIVEN AT THE END OF THE 1ST WEEK OF CLASSES.
NOTE THAT ADDITIONAL MATERIAL MAY BE INTRODUCED AND INCLUDED IN THIS TEST.

TOPICS:

(1) Solving Equations
(2) Solving Inequalities
(3) Graphs of Equations
(4) Linear Equations in Two Variables

TOPIC #1: Solving Equations

Solving Linear Equations

PRACTICE:

Solve the equations and check your solutions.

1.1 $7x+3=3x-17$

1.2 $2(x+5)-7=3(x-2)$

1.3 $\frac{x}{5}-\frac{x}{2}=3+\frac{3x}{10}$

1.4 $\frac{3}{2}(x+5)-\frac{1}{4}(x+24)=0$

Solving Quadratic Equations

You should be familiar with three methods for solving quadratic equations: (1) Solving by Factoring and (2) Solving by the Quadratic Formula and (3) Solving by square roots. All three methods require a student to use algebraic properties to manipulate the equation into standard form for a quadratic equation.

STANDARD FORM OF A QUADRATIC EQUATION

$$ax^2+bx+c=0$$

where $a\neq 0$.

NOTES:

- Many students ignore the condition that $a\neq 0$. This is necessary because were $a=0$, the equation would not be a quadratic. It would be linear.

- The values of a, b and c may be negative or positive. The "+" signs used in writing standard form, however, do not change.

- This should not be confused with Standard Form for the Equation of a Line. The "b" used in the Standard Form of a Quadratic Equation is *not* the y-coordinate of the y-intercept.

SOLVING BY FACTORING

EXAMPLE: Solve $5x^2 - 7x - 9 = {}^{-}11$.

Step 1: Place equation in standard form.

$$5x^2 - 7x + 2 = 0$$

Step 2: Factor.

$$(5x - 2)(x - 1) = 0$$

Step 3: Apply the Zero Product Rule.

> ***The Zero Product Rule***
>
> If two (or more) numbers are multiplied together such that their product is zero, then at least one of the numbers is zero.

So, here if $(5x - 2)(x - 1) = 0$, then either $(5x - 2) = 0$ or $(x - 1) = 0$.

Step 4: Solve both linear equations.

$$\begin{aligned} (5x - 2) &= 0 \\ 5x &= 2 \\ x &= \frac{2}{5} \end{aligned} \qquad \text{OR} \qquad \begin{aligned} (x - 1) &= 0 \\ x &= 1 \end{aligned}$$

You should write your solution as $x = \left\{\frac{2}{5}, 1\right\}$. **NOTE:** Use braces; do not use parentheses.

PRACTICE:

Solve the equations by factoring and check your solutions.

1.5 $x^2 + 15x + 36 = 0$

1.6 $x^2 + 5x = 6x + 12$

1.7 $3x^2 + 7x + 4 = 10$

1.8 $6x^2 + 3x = 0$

SOLVING BY THE QUADRATIC FORMULA

The Quadratic Formula

$$x = \frac{^{-}b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE: $3x^2 + 5x + 3 = 5$

Step 1: Place equation in standard form.

$$3x^2 + 5x - 2 = 0$$

Step 2: Identify a, b and c.

$a = 3; \quad b = 5; \quad c = {}^{-}2$ **NOTE:** Be sure to note **NEGATIVE** values.

Step 3: Write the quadratic formula substituting these values for a, b and c.

$$x = \frac{^{-}(5) \pm \sqrt{(5)^2 - 4(3)(^{-}2)}}{2(3)}$$

NOTE: Parentheses are recommended.

Step 4: Simplify and reduce.

$$x = \frac{^{-}(5) \pm \sqrt{(5)^2 - 4(3)(^{-}2)}}{2(3)} = \frac{^{-}5 \pm \sqrt{25+24}}{6}$$

$$= \frac{^{-}5 \pm \sqrt{49}}{6}$$

$$= \frac{^{-}5 \pm 7}{6}$$

$$= \frac{^{-}12}{6} \; OR \; \frac{2}{6}$$

Present your solution as $x = \left\{ ^{-}2, \frac{1}{3} \right\}$

NOTES:

(1) Catch the double negative.

(2) Reduce fractions and radicals.

(3) Braces.

PRACTICE:

Solve the equations by using the Quadratic Formula.

1.9 $x^2 - 10x + 22 = 0$

1.10 $2x^2 - x - 1 = 0$

1.11 $4x^2 + 4x = 7$

1.12 $8x = 5 + 2x^2$

SOLVING BY SQUARE ROOTS

EXAMPLE: $4x^2 - 48 = 0$

Step 1: With no bx term, move the c term to the other side

$$4x^2 = 48$$

Step 2: Isolate the x^2 by dividing each side by 4

$$x^2 = 12$$

Step 3: Take the square root of both sides

$$x = \pm\sqrt{12}$$

Step 4: Simplify the square root if possible

$$x = \pm 2\sqrt{3}$$

PRACTICE:

Solve the equations by extracting square roots.

1.13 $x^2 = 49$

1.14 $3x^2 = 81$

1.15 $(x-12)^2 = 16$

1.16 $(x+2)^2 = 14$

Absolute Value Equations

$$|x| = A \quad \Leftrightarrow$$
$$x = A \ \ and \ \ x = -A$$

where (1) A is any real number; and
(2) A is nonnegative.

EXAMPLE: Solve $|x-2| = 3$.

Step 1: Apply the definition.

$$x - 2 = 3 \ \ and \ \ x - 2 = -3$$

Step 2: Solve each equation.

$$x = 5 \ \ and \ \ x = -1$$

EXAMPLE: Solve $|x^2 - 3x| = -4x + 6$.

Step 1: Apply the definition.

$$x^2 - 3x = -4x + 6 \ \ and \ \ x^2 - 3x = -(-4x + 6)$$

Step 2: Solve each equation.

$$x^2 + x - 6 = 0 \ \ and \ \ x^2 - 7x + 6 = 0$$
$$x = -3, 2 \ \ and \ \ x = 1, 6$$

Step 3: Check for extraneous solutions

$$Solutions\ are \ \ x = -3, 1 \ \ (2 \ and \ 6 \ are\ extraneous)$$

Radical Equations

The steps involved in solving these types of equations are to isolate the radical or rational exponent term and apply properties of exponents to clear the radical term. Also note that extraneous solutions can be obtained.

EXAMPLE: Solve $4x^{3/2} - 32 = 0$.

Step 1: Isolate the radical term.

$$4x^{3/2} = 32$$
$$x^{3/2} = 8$$

Step 2: Use properties of exponents to clear the radical term. In this case, raise both sides to the 2/3 power.

$$x = 8^{2/3}$$

Step 3: Check this in the original equation. The solution is $x = 4$.

EXAMPLE: Solve $\sqrt{2x+7} - x = 2$.

Step 1: Isolate the radical term.

$$\sqrt{2x+7} = x + 2$$

Step 2: Use properties of exponents to clear the radical term and solve. In this case, square both sides.

$$2x + 7 = x^2 + 4x + 4$$
$$x^2 + 2x - 3 = 0$$
$$x = -3, 1$$

Step 3: Check these in the original equation. The only solution is $x = 1$.

EXAMPLE: Solve $\sqrt{2x-5}-\sqrt{x-3}=1.$

Step 1: Isolate one of the radical terms.

$$\sqrt{2x-5}=\sqrt{x-3}+1$$

Step 2: Use properties of exponents to clear the radical term you chose to isolate. In this case, square both sides.

$$2x-5=x-3+2\sqrt{x-3}+1$$
$$2x-5=x-2+2\sqrt{x-3}$$

Step 3: Now isolate the other radical term and solve. In this case, square both sides again.

$$x-3=2\sqrt{x-3}$$
$$x^2-6x+9=4(x-3)$$
$$x^2-10x+21=0$$
$$x=3,7$$

Step 4: Check these in the original equation. The solutions are $x=3,7.$

Rational Equations

The steps involved in solving these types of equations is to clear the denominator by finding the LCD (least common denominator) and multiplying every term by this LCD. Also note that extraneous solutions can be obtained.

EXAMPLE: Solve $\frac{x}{3}+\frac{3x}{4}=2$.

Step 1: Find the LCD and multiply each side by it. Then solve for the variable. In this case, the LCD is 12.

$$12\left[\frac{x}{3}+\frac{3x}{4}=2\right]$$
$$4x+9x=24$$
$$x=\frac{24}{13}$$

Step 2: Check this in the original equation. The solution is $x = \frac{24}{13}$.

EXAMPLE: Solve $\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}$.

Step 1: Find the LCD and multiply each side by it. Then solve for the variable.
In this case, the LCD is x^2-4 *or* $(x+2)(x-2)$.

$$(x-2)(x+2)\left[\frac{1}{x-2} = \frac{3}{x+2} - \frac{6x}{x^2-4}\right]$$
$$x+2 = 3(x-2) - 6x, \qquad x \neq \pm 2$$
$$x+2 = -3x-6$$
$$x = -2$$

Step 2: Check this in the original equation. In the original equation, $x = -2$ yields a denominator of zero. Therefore it is an extraneous solution, and so the original equation has NO SOLUTION.

PRACTICE:

Solve the equations and check your solutions.

1.17 $|3x+2| = 7$

1.18 $|x^2-6| = x$

1.19 $|x^2-3| = 2x$

1.20 $\sqrt{3x-2} + x = 4$

1.21 $2\sqrt{x} - 5 = x$

1.22 $\sqrt{2x+3} - 2\sqrt{x-2} = 1$

1.23 $(x-5)^{3/2} = 8$

1.24 $(x-1)^{2/3} - 25 = 0$

1.25 $3x(x-1)^{1/2} + 2(x-1)^{3/2} = 0$

1.26 $\frac{x}{5} - \frac{x}{2} = 3 + \frac{3x}{10}$

1.27 $\frac{4}{x} - \frac{5}{3} = \frac{x}{6}$

1.28 $\frac{x+1}{3}-\frac{x+1}{x+2}=0$

1.29 $\frac{6}{x}-\frac{2}{x+3}=\frac{3(x+5)}{x^2+3x}$

TOPIC #2: Solving Inequalities

Linear Inequalities

PRACTICE:

Solve the inequalities, give the solution in interval notation and graph the solution on a number line.

2.1 $2x+7<3+4x$

2.2 $\frac{3}{4}x-6\le x-7$

2.3 $1<2x+3<9$

2.4 $-1<2-\frac{x}{3}<1$

*> or < produces an open circle on the number line and parentheses in interval notation

*$\ge$ or $\le$ produces a closed circle on the number line and brackets in interval notation

*remember to flip the inequality when multiplying or dividing by a negative number

Absolute Value Inequalities:

Let x be a variable or an algebraic expression and let a be a real number such that $a \ge 0$.

1. $|x| < a \Leftrightarrow -a < x < a.$

2. $|x| > a \Leftrightarrow x < -a \quad or \quad x > a.$

These rules are also valid if $<$ or $>$ is replaced by $\le$ or $\ge$.

Note

If $a<0$:

- then the solution of $|x| < a$ is the empty set $\varnothing$ (no solution)
- then the solution of $|x| > a$ is all real numbers $\mathbb{R}$.

EXAMPLE: Solve $|x-5|<2$.

Step 1: Apply the definition.

$$-2<x-5<2$$

Step 2: Solve the resulting compound inequality.

$$3<x<7$$

The solution set is all real numbers that are greater than 3 and less than 7, which is denoted by (3,7).

EXAMPLE: Solve $|x+3|\geq 7$.

Step 1: Apply the definition.

$$x+3\leq -7 \quad or \quad x+3\geq 7$$

Step 2: Solve each inequality.

$$x\leq -10 \quad or \quad x\geq 4$$

The solution set is all real numbers that are less than or equal to -10 or greater than or equal to 4. The interval notation for this solution set is

$$(-\infty,-10]\cup[4,\infty).$$

Polynomial and Rational Inequalities:

To solve a polynomial (rational) inequality, you can use the fact that a polynomial can change signs only at its zeros. Between two consecutive zeros a polynomial must be entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real number line into intervals in which the polynomial has no sign changes. These zeros are the **critical numbers** of the inequality, and the resulting intervals are the **test intervals** for the inequality.

EXAMPLE: Solve $x^2 - x - 6 < 0.$

Step 1: By factoring the quadratic, the critical numbers are $x = -2 \quad and \quad x = 3.$.

$$x^2 - x - 6 = (x+2)(x-3) = 0$$

Step 2: So the polynomial's test intervals are

$$(-\infty, -2),\ (-2, 3),\ (3, \infty).$$

Step 3: In each test interval, choose a representative x-value and evaluate the polynomial.

Interval	x-Value	Polynomial Value	Conclusion
$(-\infty, -2)$	$x = -3$	6	Positive
$(-2, 3)$	$x = 0$	-6	Negative
$(3, \infty)$	$x = 4$	6	Positive

Step 4: The solution are interval(s) that yield a negative value since the polynomial expression is < 0.

This implies that the solution is the interval (-2,3).

EXAMPLE: Solve $\dfrac{2x-7}{x-5} \le 3.$

Step 1: Begin by writing the rational inequality in general form. (Get "zero" on one side)

$$\frac{2x-7}{x-5} - 3 \le 0$$
$$\frac{2x-7-3x+15}{x-5} \le 0$$
$$\frac{-x+8}{x-5} \le 0$$

Step 2: Set both numerator and denominator equal to zero to determine critical #'s: $x = 5,\ x = 8.$
So the test intervals are:

$$(-\infty,5),\ (5,8),\ (8,\infty).$$

Step 3: In each test interval, choose a representative x-value and evaluate the rational expression.

Interval	x-Value	Expression Value	Conclusion
$(-\infty,5)$	$x=4$	-4	Negative
$(5,8)$	$x=6$	2	Positive
$(8,\infty)$	$x=9$	-1/4	Negative

Step 4: The solution are interval(s) that make the rational expression negative or zero since it is ≤ 0.

This implies that the solution lies in the intervals $(-\infty,5)\cup[8,\infty)$.

PRACTICE:

Solve the following inequalities. Write your solution using interval notation.

2.5 $|9-2x|-2<-1$

2.6 $3|x+14|+2>17$

2.7 $|x-7|-2<-5$

2.8 $x^2+2x-3>0$

2.9 $x^2-6x+9\leq 16$

2.10 $4x^3-6x^2<0.$

2.11 $\dfrac{x+12}{x+2}\geq 3$

2.12 $\dfrac{2x-3}{x+5}-4<5$

2.13 $\dfrac{4}{x+5}>\dfrac{1}{2x+3}$

TOPIC #3: Graphs of Equations

EXAMPLE: Sketch the graph of $y = 7 - 3x$.

Step 1: Plug in values for x, and solve for y.

when $x = 0,1,2,3,4$ $y = 7,4,1,-2,-5$, respectively

Step 2: Plot the points on a coordinate plane… $(0,7),(1,4),(2,1),(3,-2),(4,-5)$.

Step 3: Draw a line that passes through the 5 coordinates.

EXAMPLE: Sketch the graph of $y = x^2 - 2$.

Step 1: Plug in values for x, and solve for y.

when $x = -2,-1,0,1,2,3$ $y = 2,-1,-2,-1,2,7$, respectively

Step 2: Plot the points on a coordinate plane… $(-2,2),(-1,-1),(0,-2),(1,-1),(2,2),(3,7)$.

Step 3: Draw a smooth curve (parabola) that passes through the 6 coordinates.

Intercepts of a Graph

1. To find x-intercepts, let y be zero and solve the equation for x.
2. To find y-intercepts, let x be zero and solve the equation for y.

EXAMPLE: Find the x- and y-intercepts of the graph of each equation.

a. $y = x$ **b.** $y = x^3 - 4x$ **c.** $y^2 = x + 4$

SOLUTION:

a. Let y = 0. Then x = 0. The graph has only one intercept, (0,0).
b. Let y = 0. Then

$0 = x^3 - 4x = x(x^2 - 4)$

has solutions x =0 and x = ± 2. Therefore, the x-intercepts are (0,0), (2,0), (-2,0).

Let x = 0. Then

$y = (0)^3 - 4(0) = 0$

Therefore, the y-intercept is (0,0).

c. Let y = 0. Then $(0)^2 = x + 4$, and $-4 = x$. Therefore, the x-intercept is (-4,0).

Let x = 0. Then $y^2 = 0 + 4 = 4$ has solutions $y = \pm 2$. Therefore, the y-intercepts are (0,2), (0,-2).

PRACTICE:

Use a table of values and intercepts to graph each equation on a separate coordinate plane.

3.1 $y = -3x + 1$

3.2 $y = 2x - 3$

3.3 $y = x^2 - 2x$

3.4 $y = -x^2 - 2x$

3.5 $y = x^3 + 3$

3.6 $y = x^3 - 1$

3.7 $y = \sqrt{x-3}$

3.8 $y = \sqrt{1-x}$

3.9 $y = |x-6|$

3.10 $y = 1 - |x|$

3.11 $x = y^2 - 1$

3.12 $x = y^2 + 5$

TOPIC #4: Linear Equations in Two Variables.

The equation of a line can be written in three forms:

Standard Form:

$$Ax + By = C$$

where (1) A, B and C are integers; and
(2) A and B aren't both zero.

Slope-Intercept Form:

$$y = mx + b$$

where (0, b) is the y-intercept.

Point-Slope Form:

$$(y - y_1) = m(x - x_1)$$

where (x_1, y_1) is any point on the line.

If you are an upperclassman, you should be familiar with all 3 forms. Despite exposure to more than one form, most students are most comfortable with Slope-Intercept Form. In fact, many students are not exposed to the other forms until high school. I agree that Slope-Intercept form is the most useful for *graphing* lines on an x-y coordinate plane. When *writing* the equation of a line, however, Point-Slope Form is more efficient.

Here is how it is done:

EXAMPLE: Write the equation of the line through $A(^{-}5, 3)$ and $B(7,1)$.

POINT-SLOPE FORM

Step 1: Find slope using $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Here, $m = \frac{1-3}{7-(^{-}5)} = \frac{^{-}2}{12} = \frac{^{-}1}{6}$.

NOTES:

- It doesn't matter which point you make point 1.
- Watch for the double negative.
- Reduce the fraction.
- Denominators of 1 should not be written.

Step 2: Plug in slope and ONE point into Point-Slope Form.

Using $m = \frac{^{-}1}{6}$ and $B(7, 1)$, we get $(y-1) = \frac{^{-}1}{6}(x-7)$

NOTES:

- Only one point should be used.
- The equation needs an "x" and a "y"

Unless instructed otherwise, this is an acceptable form for the equation of a line. Re-read the PROBLEM set forth at the top of this page. It does not specify a specific form. Use point-slope form because it is the most easily and most quickly attained. In addition, distributing and adding fractions may result in arithmetic mistakes—mistakes that will not be made if you simply leave the equation in point-slope form.

WARNING: Many students prefer to rely solely on slope-intercept form. These students plug in the slope and a point and "solve for b," the y-coordinate of the y-intercept. This process is inefficient and will not be permitted in this Calculus Course.

SLOPE-INTERCEPT FORM

If you are asked to write the equation in Slope-Intercept Form, you must perform a few more steps.

Step 3: Apply the distributive property to distribute the slope.

$$(y-1)=\frac{^-1}{6}x+\frac{7}{6}$$

NOTE: Watch for the double negative.

Step 4: Isolate y using the addition property.

$$y=\frac{^-1}{6}x+\frac{7}{6}+1$$

Step 5: Combine like terms by finding a common denominator.

$$y=\frac{^-1}{6}x+\frac{13}{6}$$

NOTE: The y-intercept is $\left(0,\frac{13}{6}\right)$.

STANDARD FORM

If you are asked to write the equation in Standard Form, you must perform a few more steps.

Step 6: Clear the denominator by multiplying ***both*** sides of the equation by the Least Common Multiple of all of the denominators.

$$6y=^-1x+13$$

NOTE: This is necessary because A, B and C must be ***integers***!

Step 7: Use the addition/subtraction property to bring the x-term to the left side.

$$1x+6y=13$$

NOTES:

- The x-term ***must*** come before the y-term.
- The coefficient of 1 need ***not*** be written.

PRACTICE:

Write the equations ***in all 3 forms*** for the lines going through the pairs of points below.

4.1 $C(5, 9)$; $D(3, {}^{-}2)$

4.2 $E({}^{-}11, 4)$; $F({}^{-}5, {}^{-}1)$

4.3 $G(24, {}^{-}20)$; $H(6, 4)$

4.4 $J(0, 3)$; $K(1, {}^{-}1)$

> ***Reminder:***
>
> Although you need not always start with point-slope form, using slope-intercept form to "solve for b" is not acceptable.

Even students who exhibit exceptional algebra skills tend to have difficulties when it comes to identifying and writing equations for horizontal and vertical lines. We will review these topics during the first few days. Nevertheless, here is a quick review.

HORIZONTAL LINES

Horizontal lines have $m = 0$. Therefore, when *m* is plugged into Point-Slope Form, the x-term goes "away"—that is, it goes to zero.

$$\begin{aligned} (y - y_1) &= 0(x - x_1) \\ (y - y_1) &= 0 \\ y &= y_1 \end{aligned}$$

Some students have a difficult time identifying horizontal lines because they mechanically apply the steps set forth above. Here is a good rule-of-thumb to save you time:

> **HORIZONTAL LINES**
>
> If both points have the same y-coordinate, the line is ***horizontal*** and the equation of that line is $y = y_1$.

Example: Find the equation of the line through $P(3, {}^{-}8)$ and $R(117, {}^{-}8)$.

While you can find slope and use point-slope form, recognizing the common y-coordinate makes the process much faster. Here the equation of the line is $y = {}^{-}8$.

If you fail to see the common y-coordinate, you will likely waste time going through the following steps:

$$m = \frac{\left(^{-}8\right)-\left(^{-}8\right)}{117-3} = \frac{0}{114}$$

$$\begin{aligned} \left(y-\left(^{-}8\right)\right) &= \frac{0}{114}(x-3) \\ y+8 &= \frac{0}{114}x-0 \\ y+8 &= 0 \\ y &= {}^{-}8 \end{aligned}$$

VERTICAL LINES

The term "slope" is undefined for vertical lines. It is best to state that vertical lines have "no slope." If a line has no ***slope***, you can't use point-slope form. There is ***NO*** slope. You can also not use slope-intercept form. There is ***NO*** slope. But there is an equation. So, what do you do?

First, you have to recognize a vertical line. Here is a good rule-of-thumb:

VERTICAL LINES

If both points have the same x-coordinate, the line is ***vertical*** and the equation of that line is $x = x_1$.

Example: Find the equation of the line through $V\left(6, {}^{-}256\right)$ and $W\left(6, 103\right)$.

Common x-coordinate $\rightarrow$ No Slope $\rightarrow$ No Point-Slope $\rightarrow$ No Slope-Intercept $\rightarrow$ No "y=mx+b"

IF NO Y=, then use X=. Here, the equation of x = 6.

PRACTICE:

Write the equations for the lines going through the pairs of points below

4.5 $S\left(2,11\right); T\left(^{-}6,11\right)$

4.6 $A\left(^{-}13,7\right); B\left(^{-}13, {}^{-}8\right)$

4.7 $J\left(6, {}^{-}27\right); K\left(6, 14\right)$

4.8 $L\left(0, 17\right); M\left(0, 2\right)$