THE PREP

St. Joseph's Preparatory School

# BC Calculus Review Packet

## Summer 2020

**Directions: Complete all the problems #1-41 on separate sheets of paper. This assignment will be collected and graded during the 1st week of school.**

**Evaluate the following limits, if they exist. If the limit does not exist, find the one-sided limits (if they exist). If a limit does not exist but is infinite, say so.**

1. $\lim_{x\to 0^+} \dfrac{\cos x}{x}$

2. $\lim_{x\to 1}\left(x^2-4x+1\right)$

3. $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right)$

4. $\lim_{x\to 0} \dfrac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$

5. $\lim_{x\to\infty} \dfrac{x^4+1}{x^3-1}$

6. $\lim_{x\to-\infty} \dfrac{e^x}{e^x+1}$

7. $\lim_{x\to 0} \dfrac{\sin 3x}{2x}$

8. $\lim_{x\to 1} \dfrac{\ln x}{x-1}$

9. $\lim_{x\to \pi/4} \dfrac{2\sin x+\sqrt{2}}{2\sin x-\sqrt{2}}$

**Suppose $f(x)$ *and* $g(x)$ are differentiable functions whose values at $x = 0, 1,$ *and* $2$ are shown in the table below:**

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $f(x)$ | 4 | 0 | 2 |
| $f'(x)$ | 0 | 1 | 4 |
| $g(x)$ | 0 | 3 | 1 |
| $g'(x)$ | 3 | 5 | 1 |

10. Find $\lim_{x\to 0} \dfrac{f\left(e^x\right)}{g(2+x)}$

11. Find $\lim_{x\to 1} \dfrac{f(x)}{g(x)-3}$

12. If $h(x) = x^2 f(x)$, find $h'(2)$.

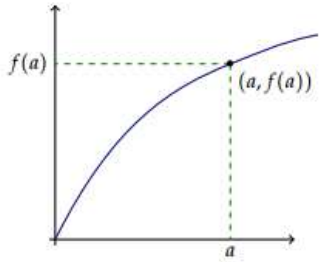
13. If $h(x) = g\left(f(2x)\right)$, find $h'(1)$.

**Consider the following function** $f(x)=\begin{cases} 2x+1 & x\le -1 \\ x^2-1 & -1<x\le 1 \\ 2x-2 & 1<x\le 2 \\ -x^2+4x-2 & x>2 \end{cases}$

14. For which $x$ is $f$ continuous?

15. For which $x$ is $f$ differentiable?

**The graph of the function $f$ is given below. It is given that $f(0)=0$. The point $x=a$ is as shown on the graph.**




16. Arrange the following quantities in ascending order, from top to bottom. (So the smallest quantity should be on the top and the largest on the bottom). Explain your reasoning.

$f'(a)$ $\quad f''(a)$ $\quad \dfrac{f(a)}{a}$ $\quad \lim\limits_{h\to 0}\dfrac{f(h)}{h}$ $\quad$ *zero*

**Find** $\dfrac{dy}{dx}$.

17. $y=3x^2+1$

18. $y=\dfrac{x}{3x^2+1}$

19. $y=\sec^3 x$

20. $y=2^{\sin^2 x+\cos^2 x-1}$

21. $y=e^{x^2\sin x}$

22. $y=\ln 3^x$

23. $xe^y=y-1$

24. $y^2=\dfrac{x}{x+1}$

25. $y=(1+1/x)^x$

26. Evaluate $f'\left(\dfrac{1}{2}\right)$, where $f(x)=\dfrac{2x+1}{2x-1}$.

27. Evaluate $f'\left(\dfrac{\pi}{4}\right)$, where $f(x)=\sin^3 x\cos x$.

28. Use the limit definition of the derivative to find $f'(x)$ for $f(x)=\dfrac{6}{x^2+1}$.

29. Find the equations of both lines through the origin which are tangent to the circle $(x-2)^2+y^2=1$.

30. Consider the curve with equation $y^2=x^2(x+1)$.

- Find the slope of the line tangent to the curve through the point (3,-6).
- Find all points on the curve to which the tangent line is *horizontal*.
- Find all points on the curve to which the tangent line is *vertical*.

31. Coffee is dripping from a conical filter into a cylindrical coffeepot. Both the filter cone and coffeepot have diameter 6 inches. The filter cone has height 6 inches as well. When the coffee in the cone is 5 inches deep, the coffee is draining at the rate of 10 cubic inches per minute.

- How fast is the level in the pot rising?
- How fast is the level in the cone falling at the same time?

32. Consider the function $f(x)=\dfrac{(x+1)^2}{x^2+1}$.

- Find the domain.
- Find all horizontal and vertical asymptotes.
- Find the intervals of increase and/or decrease.
- Find any relative maxima or minima.
- Find the intervals of concavity. Find any inflection point(s).
- Find the absolute minimum and maximum.

33. The BC Math company is manufacturing chromebooks. Suppose it costs $C(x)=4000x-100x^2+x^3$ dollars to make $x$ computers.

- Assume that no matter how many computers are produced, the market price for a computer is $2800. How many computers should we manufacture to maximize our profits?
- What is the minimum average cost of manufacturing a computer?

**Compute the following integrals. For definite integrals, your answer should be a number. For indefinite integrals, your answer should be the most general antiderivative.**

34. $\int_{-1}^{1} \left(3x^2 - 4x + 7\right) dx$

35. $\int \frac{dx}{x\sqrt{x}}$

36. $\int \sqrt{x} \sin\left(2x^{3/2}\right) dx$

37. $\int_{0}^{1/2} x^3 \left(1 + 9x^4\right)^{-3/2} dx$

38. Consider a parabolic arch whose height and width are the same length $h$. What is the area enclosed by the arch?

***Hint:*** If you put the origin in the middle of the bottom of the arch, the curve can be described by $y = h - ax^2$. computers. What is $a$?

39. The function $f(x)$ is differentiable. In the table below are some values of $f(x)$ and its derivative $f'(x)$.

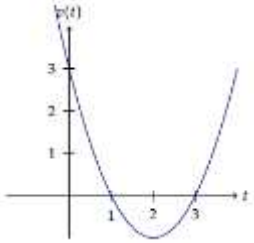
| $x$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $f(x)$ | 3 | 2 | 0 | 1 | 4 | 5 |
| $f'(x)$ | -3 | -2 | -1 | 2 | 3 | 4 |

- Approximate $\int_0^6 f(x)dx$ using three rectangles and midpoints as sample points.
- Approximate $\int_0^6 f(x)dx$ using six rectangles and right endpoints as sample points.
- Find the exact value of $\int_1^4 f'(x)dx$. Explain your steps.

40. Let $a$ be the function defined by $a(y)=\int_0^y \frac{dt}{\sqrt{1+t^2}}$ $(\triangleright)$

- Find $a'(y)$.
- Let $s(x)$ be the inverse function of $a$, so that $a(s(x))=x$ $(\dagger)$
  Use $(\triangleright)$ *and* $(\dagger)$ to show that $s'(x)=\sqrt{1+s(x)^2}$.
- Show that $s''(x)=s(x)$.

41. Imagine yourself standing alongside a road. At time $t=0$ a car passes by you traveling forward in the positive $x$ direction. The car's *velocity* $v(t)$ is given by the graph below:




Let $s(t)$ be the car's position function. Use the diagram to answer the following questions:

- Place these numbers in ascending order: $s(1), s(2), s(3)$. Justify!
- For each of the times $t=1,2,3,$ is the car moving forward or backward, or neither? It is accelerating or decelerating, or neither? Justify!